# Deterministic Dynamics and Cointegration of Higher Orders

Antoine d'Autume

UNIVERSITE DE PARIS I - PANTHEON-SORBONNE

Cahiers de Recherche
Economie, Mathématiques et Applications

# DETERMINISTIC DYNAMICS AND COINTEGRATION OF HIGHER ORDERS

Antoine D'AUTUME

92.43

CAHIERS ECOMATH - Université de Paris I
12 Place du Panthéon - 75005 Paris

# Deterministic dynamics and cointegration of higher orders

**A. d'AUTUME***

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ABSTRACT

We offer here a new method to characterize cointegration properties of higher orders. This method relies on a very simple device. We first examine the question in a deterministic framework, where cointegration properties pertain to rates of growth. We relate these properties to the structure of the Jordan matrice associated with the model. We then show how they transpose to a stochastic framework and we recover properties derived by Johansen and others

**Dynamique deterministe et coïntegration d'ordre supérieur.**

RESUME

Nous présentons une méthode nouvelle pour déterminer les propriétés de coïntégration d'ordre supérieur. Cette méthode repose sur un procédé très simple. Nous étudions d'abord le problème dans un cadre déterministe, où les propriétés ont trait aux taux de croissance. Nous relions ces propriétés à la structure de la matrice de Jordan associée au modèle puis nous montrons comment elles se transposent dans un cadre stochastique. Nous montrons enfin comment l'on retrouve simplement des propriétés établies par Johansen et d'autres auteurs.

* M.A.D., URA 926 du CNRS, Université Paris I, 90 rue de Tolbiac 75634 PARIS Cedex 13

In their pionneering analysis Engle, Granger(1987) showed how integration and cointegration properties of a vector ARMA process $G(L)\ x_t = B(L)\ \epsilon_t$ are related to the algebraic properties of the polynomial matrix G(L) and of its adjoint matrix. Although part of their analysis covers the general case, they were led to focus on the case of I(1) processes, namely of processes integrated of order one. They then stated the basic representation theorem. An I(1) process can be represented either as a VAR in variations or as an Error Correction Model, which is necessarily cointegrated. How this result can be extended to higher orders of integration remained somewhat obscure. Much work has now been devoted to the analysis of the structure of cointegrated models : see in particular Johansen(1988,1991), Yoo(1987), Gourieroux, Monfort(1990), Davidson(1991), Hylleberg, Mizon(1989), Clements(1990), Granger,Lee(1988), Gregoir, Laroque(1991). We wish however to offer an alternative method of characterization of the cointegration properties of linear models.

To this end we choose a very simple device. We first examine the question in a deterministic framework. This offers an easy way in to the question of cointegration. It provides general insights as well as a practical way to determine the cointegration structure of a model. Of course we will check that all properties apparent in the deterministic framework also hold in the stochastic sense.

We shall say that a model $G(L)\ x_t = 0$ is I(d) if it has a particular solution which is a polynomial of degree d-1 of time. It is cointegrated if a linear combination of the components this particular solution can be made to be a polynomial of degree d-2.

In principle therefore the analysis of cointegration is quite simple. In practice however it is more complicated as the general solution of linear dynamic models rests on the Jordan form of an associated matrix. Thus some algebraic complexity is unavoidable. As a matter of fact our first result is to show how special the case of an I(1) model is : it is the case where the matrix is diagonalizable. By contrast the general case of an I(d) process offers the same variety as Jordan forms. According to the number and sizes of Jordan blocks different patterns of cointegration to various orders can be obtained. We must note however that this Jordan structure may in practice be left to the background as we shall provide a direct way to determine the vectors needed to characterize the model.

The paper is organized as follows. The first section is devoted to a heuristic presentation. We derive in the second section the general solution of a deterministic model. This allows us to characterize in the third section the different cases of cointegration. The fourth section shows how this analysis transposes to a stochastic framework. This leads us to reexamine the properties of the adjoint matrix of G(L), and to relate our analysis to the results of Engle, Granger(1987). The last section is devoted to an analysis of Error Correction Models of higher orders. We show how the notion of a balanced model, introduced by Johansen(1988) fits neatly in our framework.

**1) A heuristic presentation**

We start with the deterministic model

$$G(L)\ x_t = 0 \tag{1}$$

where G(L) is a n by n polynomial matrix with degree d(G).

It is well known that a dynamic system of n equations of order d(G) is equivalent to a dynamic system of n d(G) equations of order 1. In general this last system can be solved using the Jordan form of its associated matrix. At this stage however we only note that the solution to system

(1) involves the roots $1/r_i$ of the determinant of G(L).

If all the roots of this determinant are distinct the general solution to equation (1) is

$$x_t = \sum_i h_i \; r_i^{\;t} \; v_i \qquad (2)$$

where $v_i$ is a vector belonging to the column null space of $G(1/r_i)$ and where the $h_i$ 's are scalars depending on initial conditions. If all roots lie outside the unit circle, $x_t$ tends to zero whatever the initial conditions. We shall say that the model is I(0).

The case of interest however is the case where det G(L) has a multiple unit root, say with order m. The point then is whether the matrix of the derived order one system is diagonalizable, or if one has to rely on the Jordan form. We shall show that this amounts to examining the dimension of the null space of G(1), or alternatively the rank of G(1).

If the null space of G(1) has dimension m we are in the diagonalizable case. We can choose m independent vectors $v_1, \dots, v_m$ in this column null space. If all non-unit roots are simple the solution writes

$$x_t = \sum_{i=1}^{m} h_i \; v_i \; + \; \sum_{i>1} h_i \; r_i^{\;t} \; v_i \qquad (3)$$

In general $x_t$ does not tend to zero as t tends to infinity. However $\Delta x_t$ tends to zero whatever the initial conditions. We shall say that the series is I(1).

If the null space of G(1) has a dimension smaller than m we must rely on the Jordan form and the solution involves polynomials in t. As we shall see the number and degrees of these polynomials depend on the precise structure of the Jordan matrix. If, for example, there are two blocks of size two the solution is of the following type :

$$x_t = h_1 \; v_1 + h_2 \; (t \; v_1 + v_2) + h_3 \; v_3 + h_4 \; (t \; v_3 + v_4) + + \sum_{i>1} h_i \; r_i^{\;t} \; v_i \qquad (4)$$

Vectors $v_1$ through $v_4$ will be specified later on.

The model now is I(2). $\Delta^2 x_t$ tends to zero whatever the initial conditions. Alternatively $x_t$ tends to a polynomial in t with degree one. If the variables are in logarithmic form this means that $x_t$ tends to a constant growth path, with different growth rates for different components.

In this framework cointegration is very easy to ascertain. The model is cointegrated if there exists a linear combination of the components of $x_t$ such that, whatever the initial conditions, this combination has a smaller order of integration than the series $x_t$ itself.

In the case of equation (3), that is of a I(1) model, cointegration means that there exists a row-vector $\alpha$ orthogonal to the m vectors $v_1, \dots, v_m$. Indeed, in such a case,

$$\alpha \; x_t = \sum_{i=1}^{m} h_i \; \alpha \; v_i + \sum_{i>m} h_i \; r_i^{\;t} \; \alpha \; v_i$$

$$= \sum_{i>m} h_i \; r_i^{\;t} \; \alpha \; v_i$$

which tends to zero whatever the initial conditions. As the n-dimensional vectors $v_1, \dots, v_m$ are linearly independent the Engle,Granger (1987) result obviously follows. If $m < n$ there exists n-m cointegration vectors : in Engle,Granger terminology the cointegration rank is n-m. If $m = n$ there does not exists any cointegration vector.

In the case of equation (4), which is a special case of a I(2) series, the result is obvious. $\alpha\ x_t$ is I(1) whatever the initial conditions if and only if $\alpha\ v_1 = \alpha\ v_3 = 0$. $\alpha\ x_t$ is I(0) only if moreover $\alpha\ v_2 = \alpha\ v_4 = 0$. One thus sees that the cointegration properties of the model depend in a precise way on the Jordan form of the associated matrix.

Before turning to the formal analysis a caveat is in order. Exporting the integration and cointegration definitions to the deterministic framework must not be taken too seriously. A deterministic trend is of course something very different from a stochastic one and integration and cointegration really make sense in the stochastic framework only. Our terminology will however have the advantage to make self-evident the relationship between deterministic and stochastic properties.

**2) Solution of a deterministic model**

We consider the linear dynamic system

$$G(L)\ x_t = 0 \tag{5}$$

where $x_t$ is a n-dimensional column vector and G(L) a n by n polynomial matrix with degree D. G(L) can be expanded as

$$G(L) = G_0 + G_1\ L + \ldots + G_D\ L^D$$

where coefficients $G_0, \ldots, G_D$ are square matrices. $G_0$ is assumed to be invertible and can be normalized to the n by n identity matrix I.

It is well known that system (5) is equivalent to the first order, but higher dimension, system

$$X_t = A\ X_{t-1} \tag{6}$$

where $X_t$ and the companion matrix A of G(L) are the following :

$$X_t = \begin{pmatrix} I \\ L\,I \\ \cdots \\ L^{D-1} I \end{pmatrix} x_t \qquad A = \begin{pmatrix} -G_1 & -G_2 & \cdots & \cdots & -G_D \\ I & 0 & & & 0 \\ & I & & & \\ & & & & \\ & & & I & 0 \end{pmatrix}$$

It can be checked that det(I - AL) = det G(L). Thus the roots $1/r_i$ of the determinant of G(L) are the inverses of the eigenvalues of matrix A. In particular, we may define m as the order of multiplicity of the unit root either in det G(L) or in det(I-AL).

System (6) can be solved in the usual way. There exist an invertible matrix P and a Jordan matrix J such that

$$A = P\,J\,P^{-1}$$

Matrices J, P and $P^{-1}$ have the following structures

$$J = \begin{pmatrix} \bar{J}_1 & & & \\ & \cdots & & \\ & & \bar{J}_S & \\ & & & \hat{J} \end{pmatrix} \qquad P = \begin{pmatrix} P_1 & \cdots & P_S & \hat{P} \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} Q_1 \\ \cdots \\ Q_S \\ \hat{Q} \end{pmatrix}$$

J is a block-diagonal matrix the blocks of which are Jordan. S blocks are associated with the unit root while $\hat{J}$, which is not made explicit, is the block-diagonal matrix associated with the non-unit roots. Block $\bar{J}_s$, s = 1, ... , S, has size $i_s$ and we have $\sum_{s=1}^{S} i_s = m$.

In the case of a unit eigenvalue Jordan blocks are of the following type :

$$J_1 = (1) \qquad J_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \cdots$$

With these notations $\bar{J}_s = J_{i_s}$.

To block $\bar{J}_s$ is associated a set of basis vectors $V_{s1}, \ldots, V_{s i_s}$ which are the columns of block $P_s$ in matrix P. Similarly a set of dual basis vectors $W_{s1}, \ldots, W_{s i_s}$ are the rows of block $Q_s$ in matrix $P^{-1}$. In particular, $V_{s1}$ and $W_{s i_s}$ are respectively a column and a row eigenvector associated with the unit root. The number S of Jordan blocks is therefore the dimension of the null space of A - I.

The solution of system (6) can now be derived in the usual way. Let $Y_t = P^{-1} X_t$. System (6) is equivalent to $Y_t = J\, Y_{t-1}$ the solution of which is $Y_t = J^t\, Y_0$ where $Y_0$ is a vector of arbitrary initial conditions. This implies

$$X_t = P\, J^t\, Y_0 = \sum_S P_s\, \bar{J}_s{}^t\, Y_{0s} + \hat{P}\, \hat{J}^t\, \hat{Y}_0$$

if we use block calculus and separate column vector $Y_0$ in blocks.

The powers of Jordan blocks are easy to calculate. A block of size i can be written as $J_i = I_i + N_i$ where $I_i$ is the identity matrix and $N_i$ is a nilpotent matrix with ones on the diagonal above the first diagonal and zeros elsewhere. One can check that :

$$N_i = \begin{pmatrix} 0 & 1 & & \\ & & & \\ & & & 1 \\ & & & 0 \end{pmatrix} \quad N_i{}^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & & & & \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} \quad \ldots \quad N_i{}^i = 0$$

It follows that

$$X_t = \sum_S P_s\, (I_{i_s} + N_{i_s})^t\, Y_{0s} + \hat{P}\, \hat{J}^t\, \hat{Y}_0$$

$$= \sum_{s} P_s \; (I_{i_s} + C_t^1 \; N_{i_s} + \dots + C_t^{i_s - 1} \; N_{i_s}^{i_s - 1}) + \hat{P} \; \hat{J}^t \; \hat{Y}_0$$

$$= \sum_{s} \left(V_{s\,1} \; | V_{s\,2} \; | \dots | V_{s\,i_s}\right) \begin{pmatrix} 1 & C_t^1 & & C_t^{i_s - 1} \\ & & & \vdots \\ & & & C_t^1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} Y_{0\,s\,1} \\ \hline \dots \\ \hline Y_{0\,s\,i_s} \end{pmatrix} + \hat{P} \; \hat{J}^t \; \hat{Y}_0$$

$$= \sum_{s} \; [\; V_{s\,1} \; Y_{0\,s\,1} + \left(C_t^1 \; V_{s\,1} + V_{s\,2}\right) Y_{0\,s\,2} + \dots +$$

$$+ \left(C_t^{i_s - 1} \; V_{s\,1} + \dots + C_t^1 \; V_{s\,i_s - 1} + V_{s\,i_s}\right) Y_{0\,s\,i_s} \; ] + \hat{P} \; \hat{J}^t \; \hat{Y}_0$$

Now $x_t$ is made up of the n first components of the nD-dimensional vector $X_t$. If $v_{s\,i}$ is the vector of the n first components of $V_{s\,i}$ and $\hat{p}$ the matrix of the n first rows of $\hat{P}$, we have the same formula for $x_t$ than for $X_t$ with lower case v's and p instead of upper case.

The last term could be developped similarly as the sum of terms of type $r_j{}^t \; q_j(t)$ where $1/r_j$ is a non-unit root and $q_j(t)$ is a polynomial in t. As $|r_j| < 1$ has been assumed these terms tend to zero when t tends to infinity.

Finally, it is useful to get a more direct definition of vectors $v_{s\,i}$. To provide particular solutions to system (5) these vectors must be such that

$$G(L) \left[C_t^{i_s - 1} \; v_{s\,1} + \dots + v_{s\,i_s}\right] = 0 \quad \forall t$$

As shown in the appendix this yields a simple characterization of vectors $v_{s\,i}$. In particular $\forall s$, $v_{s\,1}$ belongs to the column null-space of G(1). Moreover these vectors $v_{s\,1}$ are independent so that the number S of Jordan blocks is the dimension of the null-space of G(1), that is S = n-r where r is the rank of G(1).

The characterization of vectors $v_{s\,i}$ relies on the

Taylor expansion of the polynomial matrix G(L).

$$G(L) = G_0^* + G_1^* \Delta + \ldots + G_D^* \Delta^D$$

The coefficients in this expansion are

$$G_h^* = (-1)^h \frac{G^{(h)}(1)}{h\,!}$$

where $G^{(h)}(L)$ is the h-th derivative of G(L).

**Proposition 1**

*Consider the n-dimensional linear dynamic system* $G(L)\,x_t = 0$. *Let m be the order of multiplicity of the unit root in det* G(L). *Let* $S \leq m$ *be the dimension of the null space of* G(1). *The general solution of the system is the following :*

$$x_t = \sum_{s=1}^{S} [h_{s1} v_{s1} + h_{s2} (t\, v_{s1} + v_{s2}) + \ldots +$$

$$+ h_{s i_s} \left( C_t^{i_s - 1} v_{s1} + \ldots + C_t^0 v_{s i_s} \right) ] + \sum_{j>1} r_j^{\,t} q_j(t)$$

*where* $\sum_s i_s = m$, $q_j(t)$ *is a polynomial in t, and the* $h_{si}$ *are arbitrary scalars.*

*For given s, vectors* $v_{si}$ *are such that*

$$\begin{pmatrix} G_0^* & & & & & & \\ G_1^* & G_0^* & & & & & \\ G_2^* - G_1^* & G_1^* & G_0^* & & & & \\ G_3^* - 2G_2^* + G_1^* & G_2^* - G_1^* & G_1^* & G_0^* & & & \\ & \cdots & & & \cdots & & \\ m_{f\,0} & m_{f\,1} & m_{f\,2} & \cdots & m_{f\,f} & & \\ & \cdots & & & \cdots & & \\ m_{i_s-1\;0} & & & \cdots & & & m_{i_s-1\;i_s-1} \end{pmatrix} \begin{pmatrix} v_{s\,1} \\ v_{s\,2} \\ \\ \\ v_{s\,i_s} \end{pmatrix} = 0$$

*where* $m_{0\,0} = G_0^*$

$$m_{f\,0} = \sum_{k=0}^{f-1} (-1)^k \; C_{f-1}^{f-k-1} \; G_{f-k}^* \qquad \forall f > 0$$

*and* $m_{f\,j} = m_{f-j\;0}$

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The number of combinations $C_t^h$ is a polynomial in t of degree h. Thus $x_t$ appears as the sum of polynomials of degree $i_s$-1, and therefore as a polynomial of degree d-1, to which is added a term associated with the non-unit roots which tends to zero as t tends to infinity.

**3) Cointegration in a deterministic framework**

We shall say that a deterministic model is integrated of order d, denoted I(d), if $\Delta^d x_t$ tends to zero as t tends to infinity, whatever the initial conditions, and if $\Delta^{d-1} x_t$ does not.

The model is cointegrated of order i if there exists a row vector $\alpha$ such that $\alpha\ x_t$ is at most I(d-i). The number of such independent $\alpha$ vectors is the rank of cointegration to the order i, which will be denoted by $k_i$. Let b be the maximum degree of cointegration. The dynamic properties of a model can be summed up by the following notation : we shall say that the model is $CI(d, b\ ; k_1, \ldots, k_b)$. Obviously $k_1 \geq k_2 \geq \ldots \geq k_b > 0$.

One should note that, contrary to usual definitions, we do not exclude from the definition of cointegration the case where the cointegrating vector has a unique non-zero component. This clearly is a degenerate case where one of the initial variables is not I(d). In such a case of course the term "cointegration" becomes rather inadequate.

Integration and cointegration properties can easily be read from the form of the solution as it is given in proposition 1. This is so because the vectors $v_{si}$ in this proposition which are associated with polynomials $C_t^i$ of degree i larger than a given number span a vector subspace which is independent of the precise choice of the basis vectors. This property follows from the theory of Jordan forms and we shall recover it in a moment. Let us define define these subspaces with the help of some additional notation.

$\mathcal{L}\{v_1, \ldots, v_n\}$ denotes the vector subspace spanned by vectors $v_1, \ldots, v_n$. Let $\sigma_i = \{s \ / \ i_s = i\}$ and let $u_i$ be the number of elements in $\sigma_i$ , that is the number of Jordan blocks of size i. Let

$$F_1 = \mathcal{L}\{v_{s1} \ / \ s \in \sigma_d\}$$

$$\ldots$$

$$F_i = \mathcal{L}(\{v_{sj} \ / \ j \leq i\ ,\ s \in \sigma_d\} \cup \{v_{sj} \ / \ j \leq i-1\ ,\ s \in \sigma_{d-1}\}$$

$$\cup \ldots \cup \{v_{si} \ / \ j = 1\ ,\ s \in \sigma_{d-i+1}\}) \qquad (7)$$

$$\ldots$$

$$F_d = \mathcal{L}\{v_{sj} \ / \ \forall s, \forall j\}$$

$$F_1 \subset F_2 \subset \ldots \subset F_d$$

$F_i$ simply is the subspace spanned by vectors associated to powers of t equal to d-i or larger.

Let $f_i$ be the number of vectors which span $F_i$. Then

$$u_1 + u_2 + \dots + u_d = S$$

$$f_1 = u_d$$
$$\dots$$
$$f_i = i\, u_d + (i-1)\, u_{d-1} + \dots + u_{d-i+1}$$
$$\dots$$
$$f_d = d\, u_d + (d-1)\, u_{d-1} + \dots + u_1 = m$$

$F_1$ is spanned by $f_1$ vectors $v_{s1}$ belonging to the null-space of G(1). These vectors are independent and therefore $\dim F_1 = f_1$. For i larger than one, to the contrary, $\dim F_i$ generally differs from $f_i$, the number of basis vectors which generate $F_i$. The basis vectors $V_{si}$ are independent in the nD-dimensional space but the $v_{si}$ formed with their n first components obviously are not, as they are nD in number and belong to a n-dimensional space.

Let us now come back to the determination of vectors $v_{si}$ and of subspaces $F_i$.

Consider matrix

$$M = \begin{pmatrix} G_0^* & & & & \\ G_1^* & G_0^* & & & \\ G_2^* - G_1^* & G_1^* & G_0^* & & \\ & \dots & & \dots & \\ m_{f0} & m_{f1} & m_{f2} & \dots & m_{ff} \\ & \dots & & & \dots \end{pmatrix}$$

and let $M_i$ be the matrix made up with the i first rows and columns of M. Let $E_i$ be the null-space of matrix $M_i$. A vector $(v_1, \dots, v_i)^{tr}$ in $E_i$ is the product of i vectors with n components.

It is clear that if $(v_1, \dots, v_i)^{tr}$ belongs to $E_i$, then vector $(0, v_1, \dots, v_i)^{tr}$ belongs to $E_{i+1}$, vector $(0, 0, v_1, \dots, v_i)^{tr}$ to $E_{i+2}$ and so on. Thus we have

$$\dim E_1 < \dim E_2 < \dots < \dim E_d = \dim E_{d+1} = m$$

The inequalities are strict and the first equality encountered determines d which will be the order of integration. We know from the previous Jordan form analysis that the dimension of space $E_d$ has to be equal to the order of multiplicity of the unit-root.

Let $F_i$ be the projection of $E_d$ on the space of the i-th group of components. These $F_i$ are the subspaces already defined.

We are now ready for a straightforward analysis of integration and cointegration properties.

The order of integration is $d = \max_s i_s$. Obviously $d \leq m$. If r is the rank of G(1) the number of Jordan blocks is $S = n\text{-}r \leq m$.

A row vector $\alpha$ is a cointegrating vector at order i iff it is orthogonal to $F_i$. It exists iff $\dim F_i \leq n$ and the cointegration rank then is $k_i = n - \dim F_i$. In particular there exists a cointegration vector at order one iff $\dim F_1 = f_1 = u_d < n$. As $u_d \leq S = n - r$, we see that $k_1 = n - u_d \geq r$. Thus the rank of cointegration at order one is no smaller than the rank of G(1). The cointegration rank at orders higher than one is more difficult to ascertain as it depends on the dependence or independence of vectors in $F_i$.

Let us illustrate the analysis with an example. The parameters which are easy to determine are the order of multiplicity m and the dimension $S = n-r$ of the null-space of $G(1)$ and we know that $S \leq m$.

Let us for instance assume that $m = 6$ and $S = 3$. Six basis vectors have to be allocated to three blocks and three cases occur. Either the blocks have sizes 4, 1, 1 or 3, 2 ,1 or 2, 2, 2. Let us analyze the second case. The Jordan basis has the following structure :

$$\bar{p} = (v_{11},\ v_{12},\ v_{13} \mid v_{21},\ v_{22} \mid v_{31})$$

A basis for space $E_3$ is the following :

$$\begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} \begin{pmatrix} 0 \\ v_{11} \\ v_{12} \end{pmatrix} \begin{pmatrix} 0 \\ v_{21} \\ v_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_{21} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_{31} \end{pmatrix}$$

$$F_1 = \mathcal{L}(v_{11}) \qquad F_2 = \mathcal{L}(v_{11},\ v_{21},\ v_{21})$$

$$F_3 = \mathcal{L}(v_{11},\ v_{12},\ v_{13},\ v_{21},\ v_{22},\ v_{31})$$

The general solution is

$$x_t = h_1 \left(C_t^2\ v_{11} + t\ v_{12} + v_{13}\right) + h_2\ (t\ v_{11} + v_{12}) + h_3\ v_{11} + h_4\,(t\ v_{21} + v_{22}) + h_5\ v_{21} + h_6\ v_{31}$$

The model is $I(3)$. $\text{Dim } F_1 = 1$, $\dim F_2 \leq 3$, $\dim F_3 \leq 6$. If $n$ is large enough, the model is cointegrated to the order 1 with rank $n - 1$, to the order 2 with rank $n - \dim F_2$ and to the order 3 with rank $n - \dim F_3$ .

**4) Cointegration in a stochastic framework**

We now check that the analysis of cointegration in a deterministic framework is consistent with the usual analysis which of course takes place in a stochastic framework. This will lead us to examine what properties of the adjoint matrix of $G(L)$ can be derived from the Jordan form of the companion matrix A.

The model now is

$$G(L)\ x_t = B(L)\ \epsilon_t \tag{8}$$

$B(L)$ is a n by n polynomial matrix and $\epsilon_t$ a n-dimensional white noise. We assume that the roots of det $G(L)$ are outside the unit circle or equal to one and that the roots of det $B(L)$ all are outside the unit circle. We thus assume $B(1)$ to have full rank. This restriction will be disposed of later on.

Let us recall the standard analysis.

Let $G^a(L)$ be the adjoint matrix of $G(L)$. We know that

$$\det G(L) = (1 - L)^m\ g(L), \qquad g(1) \neq 0$$

The adjoint matrix generally has a unit-root and we state

$$G^a(L) = (1 - L)^u\ H(L), \qquad H(1) \neq 0$$

As

$$G(L)\ G^a(L) = G^a(L)\ G(L) = \det G(L)\ I$$

or

$$G(L)\ H(L) = H(L)\ G(L) = (1 - L)^{m-u}\ g(L)\ I \tag{9}$$

we get

$$(1-L)^{m-u}\ g(L)\ x_t = H(L)\ B(L)\ \epsilon_t \tag{10}$$

As $B(1)$ has full rank $H(1)B(1) \neq 0$ , so that $x_t$ is $I(d)$ with $d = m - u$.

Moreover

$$(1-L)^d \; g(L) \; \alpha \; x_t = \alpha \; H(L) \; B(L) \; \epsilon_t$$

The model is cointegrated to the order i iff $(1-L)^i$ can be factored out of $\alpha \; H(L) \; B(L)$, namely if $\alpha \; H(L)$ has a unit root with order of multiplicity equal to i.

The next proposition will show the consistency of the definitions of integration and cointegration in stochastic and deterministic frameworks. It derives from a development of the adjoint matrix $G^a(L)$.

Let

$$p = (\bar{p} \mid \hat{p}) = \left(p_1 \mid \dots \mid p_s \mid \hat{p}\right) \quad \text{with} \quad p_s = \left(v_{s\,1} \mid \dots \mid v_{s\,i_s}\right)$$

be the $(n\;,\;nD)$ matrix formed with the n first rows of matrix P. Let

$$q = \begin{pmatrix} \bar{q} \\ \hline \hat{q} \end{pmatrix} = \begin{pmatrix} q_1 \\ \hline \dots \\ \hline q_s \\ \hline \hat{q} \end{pmatrix} \quad \text{with} \quad q_s = \begin{pmatrix} w_{s\,1} \\ \hline \dots \\ \hline w_{s\,i_s} \end{pmatrix}$$

be the $(nD\;,\;n)$ matrix formed with the n first columns of matrix $P^{-1}$.

The Jordan form of the companion matrix A is

$$J = \begin{pmatrix} \bar{J} & \\ & \hat{J} \end{pmatrix}$$

$\bar{J}$ is the block-diagonal matrix associated with the unit root while $\hat{J}$ is associated with the non unit-roots.

As we have seen $\det G(L) = \det (I - A L)$ and therefore

$$\det G(L) = \det(I-JL) = \det(I-\bar{J}L)\,\det(I-\hat{J}L) = (1-L)^m \det(I-\hat{J}L)$$

$$= (1-L)^m\, g(L) \qquad \text{with} \quad g(1) \neq 0$$

Moreover $\bar{J} = I + N$ with

$$N = \begin{pmatrix} \bar{N}_1 & & \\ & \ddots & \\ & & \bar{N}_s \end{pmatrix} = \begin{pmatrix} N_{i_1} & & \\ & \ddots & \\ & & N_{i_s} \end{pmatrix}$$

$N$ is a nilpotent block-diagonal matrix with blocks of size $i_s$. The largest block has size d and therefore $N^d = 0$.

**Proposition 2**

*Let d be the maximum size of the Jordan blocks in the Jordan form of matrix A. Let the sets of vectors* $E_i$ *be defined as in (7) . Then*

i) $G^a(L) = (1-L)^{m-d}\, H(L) \quad , \quad H(1) \neq 0$

$$H(L) = g(L)\,\bar{p}\,[(1-L)^{d-1} I + (1-L)^{d-2} NL + \dots + (NL)^{d-1}]\,\bar{q} \; +$$

$$+ (1-L)^m\, \hat{p}\, (I-\hat{J}L)^a\, \hat{q}$$

ii) $\alpha H(L)$ *has a unit root with order of multiplicity equal to i iff* $\alpha$ *is orthogonal to all vectors in* $F_i$ .

The proposition shows, in a stochastic framework, that d is indeed the degree of integration of the model and that the cointegrating vectors, to the order i, are the vectors orthogonal to $F_i$ .

The first part of the proposition is proved in the appendix. The second part follows from an analysis of the developped form of H(L) given in the first part. We prove it here for $i = 1$. The proof for higher orders is given in the appendix.

We first note that

$$H(1) = g(1)\ \bar{p}\ N^{d-1}\ \bar{q} \tag{11}$$

so that $\alpha$ H(L) has a unit root iff

$$\alpha\ \bar{p}\ N^{d-1}\ \bar{q} = 0$$

Now

$$H(1) = g(1)\ (p_1\ |\ \dots\ |\ p_S) \begin{pmatrix} N_{i_1} & & \\ & \ddots & \\ & & N_{i_S} \end{pmatrix}^{d-1} \begin{pmatrix} q_1 \\ \hline \dots \\ \hline q_S \end{pmatrix}$$

$$= g(1) \sum_{s} p_s\ N_{i_s}^{\ d-1}\ q_s = g(1) \sum_{s\ \in\ \sigma_d} p_s\ N_d^{\ d-1}\ q_s$$

(as $s \notin \sigma_d \Rightarrow i_s < d$ and therefore $N_{i_s}^{\ d-1} = 0$ )

$$= g(1) \sum_{s\ \in\ \sigma_d} v_{s1}\ w_{s\, i_s}$$

(as the only non-zero entry of $N_d^{\ d-1}$ is in the first row and in the d-th, or $i_s$-th, column).

Vectors $v_{s1}$ and $w_{s\,i_s}$ , for $s \in \sigma_d$, belong to the column and row null spaces of G(1). For any row vector $\alpha$

$$\alpha\ H(1) = g(1) \sum_{s\ \in\ \sigma_d} (\alpha\ v_{s1})\ w_{s\, i_s}$$

Vectors $w_{s\,i_s}$ are independent. Thus $\alpha\, H(1) = 0$ iff $\alpha$ is orthogonal to all vectors in $E_1$.

Let us now sum up our results and relate them to the literature.

Lemma 1 in Engle-Granger(1987) basically states that

$$m \geq n - r \qquad\qquad \text{rank } H(1) \leq n - r$$

which implies that the cointegration rank at order one is not smaller than r.

These properties receive a deeper interpretation in our analysis. The first inequality is simply the well-known property relating the dimension of an eigen-subspace to the order of multiplicity of the eigenvalue. The second one expresses the fact that cointegrating vectors are orthogonal to the (projected) eigenvectors belonging to Jordan blocks of maximum size. Obviously the number of such eigenvectors is inferior to the total number $S = n - r$ of eigenvectors.

After stating their lemma Engle-Granger(1987) focused on the case $d = 1$ and recognized that in such a case the order of cointegration is equal to r, thus showing that a set of I(1) series are not cointegrated iff $r = 0$, that is iff $1 - L$ can be factored out of matrix G(L).

Our analysis shows how special the case I(1) is. It is indeed the case when the companion matrix A is diagonalizable and when all blocks therefore have size one. Thus $m = n - r$ is a necessary and sufficient for $x_t$ in model (8) to be I(1). Then rank $H(1) = n - r$. This property is proved in particular by Davidson(1991). Note that this last equality, that is the fact that the rank of cointegration at order one is equal to r, simply means that all Jordan blocks have the same size and it is not characteristic of the case I(1) only.

Engle-Granger(1987) analysis of the rank of $H(1)$ started from equality

$$G(1)\ H(1) = H(1)\ G(1) = 0 \quad \text{if } d \geq 1 \tag{12}$$

which follows from (9), with $d = m - u$. They noted that this implies that the columns of $H(1)$ belong to the null space of $G(1)$ and therefore that rank $H(1) \leq n - r$.

This property can be given a more precise form. First we may write

$$G_0^* = G(1) = C\ A$$

where $C$ and $A$ are $(n,r)$ and $(r,n)$ matrices, both having full rank. The columns of $C$ and the lines of $A$ , respectively, are two basis of the subspaces spanned by the columns and the lines of $G(1)$.

On the other hand, let $\tilde{P}_1$ and $\tilde{Q}_1$ be the $(n,u_d)$ and $(u_d,n)$ matrices formed respectively with vectors $v_{s1}$ and $w_{s\,i_s}$ , for $s \in \sigma_d$. Then

$$H(1) = g(1)\ \tilde{P}_1\ \tilde{Q}_1$$

where $g(1)$ is a non-zero scalar and matrices $\tilde{P}_1$ and $\tilde{Q}_1$ have full rank.

Relation (12) expresses orthogonality conditions between the lines and columns of $G(1)$ and $H(1)$, which clearly amount to the following conditions

$$A\ \tilde{P}_1 = 0 \qquad \tilde{Q}_1\ C = 0$$

This property is well-known and was stated by Engle-Granger(1987) for the case $I(1)$. In such a case $u_d = n - r$. Then the lines of $A$, or equivalently of $G(1)$, span the complete cointegration subspace. This is not so in general, as the lines of G(1) are orthogonal to all eigenvectors $v_{s1}$ whereas cointegration vectors need only be orthogonal to the eigenvectors belonging to Jordan blocks of maximum size.

A similar property holds for higher orders of cointegration. First note that, from proposition 2, $\alpha$ H(L) has a unit root with order of multiplicity not smaller than i iff

$$\alpha \bar{p} N^{d-1} \bar{q} = \alpha \bar{p} N^{d-2} \bar{q} = \ldots = \alpha \bar{p} N^{d-i} \bar{q} = 0$$

It is shown in the appendix that

$$\bar{p} N^{d-i} \bar{q} = \tilde{P}_i \tilde{Q}_i$$

where the columns of $\tilde{P}_i$ are the vectors which span $F_i$. Obviously a vector orthogonal to $F_i$ is a cointegration vector of order i. However matrices $\tilde{P}_i$ and $\tilde{Q}_i$ do not have full rank and some care must be taken in proving the reciprocal.

**5) Error correction models**

Let us start with a remark about the determination of vectors $v_{si}$ as it is spelled out in proposition 1. Vectors $v_{s1}$ and $v_{s2}$ for instance must be such that

$$\begin{cases} G_0^* v_{s1} = 0 \\ G_1^* v_{s1} + G_0^* v_{s2} = 0 \end{cases} \tag{13}$$

If the column null-spaces of $G_0^*$ and $G_1^*$ have a non-empty intersection one can take $v_{s1}$ and $v_{s2}$ both equal to a vector v in this intersection. Then $x_t = (h_1 + h_2 t) v$ is solution to $G(L) x_t = 0$ for all scalars $h_1$ and $h_2$. The model has at least one block of size equal to two or larger, and

$v \in F_{d-1} \subset F_d$.

This obviously generalizes. A vector belonging to $\text{Ker } G_0^* \cap \ldots \cap \text{Ker } G_{i-1}^*$ may be taken as the leading vector $v_{s1}$ of a Jordan block with size at least equal to i. Let

$$m_i = \dim \text{Ker } G_0^* \cap \ldots \cap \text{Ker } G_{i-1}^*$$

and recall that $u_i$ is the number of Jordan blocks with size i. Then

$$\text{Ker } G_0^* \cap \ldots \cap \text{Ker } G_{i-1}^* \subset F_{d-i+1} \tag{14}$$

$$u_1 + \ldots + u_d = m_1 = S$$

$$u_i + u_{i+1} + \ldots + u_d \geq m_i \quad , \quad i = 2, \ldots, d \tag{15}$$

Let us now consider the most general definition of an ECM, simply based on the Taylor expansion of matrix G(L) :

$$G_0^* \; x_t + G_1^* \; \Delta x_t + \ldots + G_j^* (L) \; \Delta^j x_t = B(L) \; \epsilon_t \tag{16}$$

This is the definition used by Johansen(1988). A more usual definition of an ECM has lags in all error terms and is typically based on the following different expansion of matrix G(L)

$$L\tilde{G}_0 \; x_t + L\tilde{G}_1 \; \Delta x_t + \ldots + L\tilde{G}_{j-1} \; \Delta^{j-1} x_t + \tilde{G}_j (L) \; \Delta^j \; x_t = B(L) \; \epsilon_t$$

It is easy however to go from one definition to the other and we stick to the simplest one. We are interested in the statistical properties of the different terms which appear in the ECM, and we state :

**Definition** : ECM (16) is satisfactory if $x_t$ is integrated of order j and if each term in the sum in the L.H.S. is stationary.

**Definition** : The model is balanced if $\sum_{i=1}^{\infty} m_i = m$, where m is the order of multiplicity of the unit root.

**Proposition 3:** A model can be represented as a satisfactory ECM iff it is balanced.

This proposition of Johansen(1988) can easily be proved within our framework. Indeed both properties, satisfactory or balanced, are easily seen to be equivalent to the fact that each Jordan block may be made up with a unique vector $v_{s1} = v_{s2} = \ldots = v_{si_s}$.

Summing relations (15) implies that

$$m = u_1 + 2u_2 + \ldots + du_d \geq \sum m_i$$

The model therefore is balanced iff relations (15) are equalities ie if

$$u_i + \ldots + u_d = m_i \qquad \forall i$$

This implies that

$$u_i = m_i - m_{i+1}$$

and that

$$\text{Ker } G_0^* \cap \ldots \cap \text{Ker } G_{i-1}^* = F_{d-i+1} \qquad \forall i$$

On the other hand let us consider a satisfactory ECM. In a deterministic framework $\Delta^{i-1} x_t$ is a combination of vectors belonging to $F_{d-i+1}$. Thus $G_{i-1}^* \Delta^{i-1} x_t$ is zero iff $F_{d-i+1} \subset \text{Ker } G_{i-1}^*$.

Therefore a model is a satisfactory ECM iff

$$F_1 \subset F_2 \subset \ldots \subset F_{d-i+1} \subset \text{Ker } G^*_{i-1} \qquad \forall i$$

or iff

$$F_{d-i+1} \subset \text{Ker } G^*_0 \cap \ldots \cap \text{Ker } G^*_{i-1} \qquad \forall i$$

or, from (14)

$$F_{d-i+1} = \text{Ker } G^*_0 \cap \ldots \cap \text{Ker } G^*_{i-1} \qquad \forall i$$

Thus a model is a satisfactory ECM iff it is balanced. It may have Jordan blocks of different sizes but each block must be made up with only one vector.

**5) The case of a unit root in the right hand side**

Our analysis rested on the assumption that matrix $B(L)$ has full rank, but it can be extended to the case where $B(L)$ has a unit root. We then premultiply the model by a polynomial matrix to obtain the following representation:

$$\hat{G}(L)\, x_t = \Delta^v \quad \hat{B}(L)\, \epsilon_t \tag{17}$$

$$v \geq 0 \qquad \hat{B}(1) \text{ of full rank}$$

Such a representation is obtained, with a diagonal $\hat{B}(L)$, if we premultiply with $B^a(L)$. But premultiplying by a matrix with lower degree is usually possible.

The previous analysis can be applied. If matrix $\hat{G}(L)$ is such that model $\hat{G}(L)\, x_t = \hat{B}(L)\, \epsilon_t$ is I(d), then model (17) is I(d-v) with the same vectors of cointegration.

**Conclusion**

We have shown how the analysis of deterministic trends can be used as a simple method to describe the cointegration properties of ARIMA models. We thus gain a clear picture of cointegration at any order. More practically we showed how all relevant properties can be derived from an analysis of a stacked matrix involving in a very specific way the coefficients of the Taylor expansion of G(L).

We did not consider multicointegration, involving "dynamic" cointegration vectors of the form $\alpha + \beta\Delta + \ldots$ While it is clear that our method can be used to determine such vectors, it is left to further research to relate it to the general ECM models put forward in particular by Johansen(1988), Granger,Lee(1988), Davidson(1991) and Gregoir,Laroque(1991).

Finally we note that the analysis of deterministic trends has an interest per se. A standard macroeconomic model would appear as $G(L)\ x_t = B(L)\ \epsilon_t + b$ , with a vector b of constants in the R.H.S. Such a model usually has a deterministic trend and we wish to know for instance whether the cointegration vectors cancel the deterministic trends. Such an analysis follows directly from the methods in this paper and the reader is referred to d'Autume(1992) for developments and illustrations.

## Appendix

**Proof of proposition 1**

We want to characterize vectors $v_1, \ldots, v_{i+1}$ such that

$$G(L)\ \left(C_t^i\ v_1\ +\ C_t^{i-1}\ v_2\ +\ \ldots\ +\ C_t^0\ v_{i+1}\right)\ =\ 0 \qquad \forall t$$

By convention $C_t^i = 0$ if $i < 0$. We shall use repeatedly the fact that

$$C_t^i = C_{t-1}^i + C_{t-1}^{i-1}$$

This implies

$$\Delta\ C_t^i = C_{t-1}^{i-1} \quad \text{and} \quad \Delta^h\ C_t^i = C_{t-h}^{i-h} \tag{18}$$

It also implies the following equality which can be proved by induction

$$C_{t-h}^s = \sum_{k=0}^{s} (-1)^k\ C_{k+h-1}^{h-1}\ C_t^{s-k} \qquad h \geq 1 \tag{19}$$

Now

$$G(L)\ \left(C_t^i\ v_1 + C_t^{i-1}\ v_2 + \dots + C_t^0\ v_{i+1}\right)$$

$$= \sum_{h=0}^{i} G_h^*\ \Delta^h \sum_{j=0}^{i} C_t^{i-j}\ v_{j+1}$$

$$= \sum_{h=0}^{i} \sum_{j=0}^{i} C_{t-h}^{i-j-h}\ G_h^*\ v_{j+1}$$

(from (18))

$$= \sum_{j=0}^{i} C_t^{i-j}\ G_0^*\ v_{j+1} +$$

$$\sum_{h=1}^{i} \sum_{j=0}^{i} \sum_{k=0}^{i-j-h} (-1)^k\ C_{k+h-1}^{h-1}\ C_t^{i-j-h-k}\ G_h^*\ v_{j+1}$$

(from (19))

$$= \sum_{f=0}^{i} G_0^*\ v_{f+1}\ C_t^{i-f} +$$

$$\sum_{h=1}^{i} \sum_{j=0}^{i} \sum_{f=j+h}^{i} (-1)^{f-j-h}\ C_{f-j-1}^{h-1}\ G_h^*\ v_{j+1}\ C_t^{i-f}$$

(letting $f = j + h + k$ )

$$= \sum_{f=0}^{i} G_0^* \quad v_{f+1} \quad C_t^{i-f} \; +$$

$$\sum_{f=1}^{i} \sum_{j=0}^{f-1} \sum_{h=1}^{f-j} (-1)^{f-j-h} \quad C_{f-j-1}^{h-1} \quad G_h^* \quad v_{j+1} \; C_t^{i-f}$$

$$= \left[G_0^* \; v_1\right] \; C_t^i \; + \sum_{f=1}^{i} \left[G_0^* \; v_{f+1} \; + \sum_{j=0}^{f-1} \sum_{h=1}^{f-j} (-1)^{f-j-h} \; C_{f-j-1}^{h-1} \; G_h^* \; v_{j+1}\right] C_t^{i-f}$$

This last expression must be zero for all t. As $C_t^{i-f}$ is a polynomial in t with degree i-f, the $C_t^{i-f}$'s, for different f, are independent functions of t and the terms into brackets must all be zero.

Let $m_{fj}$ be the coefficient of $v_{j+1}$ in the brackets associated with $C_t^{i-f}$.

$$m_{ff} = G_0^* \qquad \forall f \geq 0$$

$$m_{fj} = \sum_{h=1}^{f-j} (-1)^{f-j-h} \; C_{f-j-1}^{h-1} \quad G_h^* \; = \sum_{k=0}^{f-j-1} (-1)^k \; C_{f-j-1}^{f-j-k-1} \; G_{f-j-k}^*$$

for $j < f$

Note that $m_{nj}$ only depends on n-j.

Vectors $v_j$ must satisfy conditions

$$\sum_{j=0}^{f} m_{fj} \quad v_{j+1} \quad = 0 \qquad\qquad f = 0, 1, \ldots$$

These conditions are written in matrix form in proposition 1.

**Proof of proposition 2**

$$G(L) = G_0 + G_1 \; L + \ldots + G_D \; L^D$$

$$A = \begin{pmatrix} -G_1 & -G_2 & \cdots & \cdots & -G_D \\ I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \end{pmatrix}$$

Let

$$U(L) = \begin{pmatrix} I & & & \\ L\,I & I & & \\ \vdots & & \ddots & \\ L^{D-1}I & L^{D-2}I & \cdots & I \end{pmatrix} \quad \text{and} \quad U^{-1}(L) = \begin{pmatrix} I & & & \\ -LI & I & & \\ & \ddots & \ddots & \\ & & -LI & I \end{pmatrix}$$

It is easy to check that $U(L)$ and $U^{-1}(L)$ are indeed inverse matrices and that their determinants are equal to one. Also

$$(I-AL)\ U(L) = \begin{pmatrix} I+G_1L & G_2L & \cdots & & G_DL \\ -LI & I & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -LI & I \end{pmatrix} U(L) =$$

$$= \begin{pmatrix} G(L) & G_2L+G_3L^2+\ldots+G_DL^{D-1} & \cdots & G_DL \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} \overset{\text{def}}{=} \tilde{G}(L)$$

Taking determinants shows that

$$\det(I-AL) = \det(I-AL)\ \det U(L) = \det \tilde{G}(L) = \det G(L)$$

Let us examine the adjoint matrix $\tilde{G}(L)^a$. It is easy to check that it has $G(L)^a$ as upperleft block.

As

$$\tilde{G}(L)^a = U(L)^a\ (I-AL)^a = U(L)^{-1}\ (I-AL)^a = U(L)^{-1}\ P(I-JL)^a\ P^{-1},$$

$$G(L)^a = (I_n \mid 0 \mid \dots \mid 0)\ U(L)^{-1}\ P(I-JL)^a\ P^{-1} \begin{pmatrix} I_n \\ \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix} =$$

$$= (I_n \mid 0 \mid \dots \mid 0)\ P(I-JL)^a\ P^{-1} \begin{pmatrix} I_n \\ \overline{0} \\ \vdots \\ \overline{0} \end{pmatrix} = p\ (I-JL)^a\ q$$

Now for any block-diagonal matrix

$$M = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix} \qquad M^a = \begin{pmatrix} (\det M_2)\ M_1{}^a & \\ & (\det M_1)\ M_2{}^a \end{pmatrix}$$

Therefore

$$(I-JL)^a = \begin{pmatrix} I-\bar{J}L & \\ & I-\hat{J}L \end{pmatrix}^a = \begin{pmatrix} \det(I-\hat{J}L)\quad (I-\bar{J}L)^a & \\ & \det(I-\bar{J}L)\quad (I-\hat{J}L)^a \end{pmatrix} =$$

$$= \begin{pmatrix} g(L)\ (I-\bar{J}L)^a & \\ & (1-L)^m\ (I-\hat{J}L)^a \end{pmatrix}$$

$$G(L)^a = (\bar{p} \mid \hat{p}) \begin{pmatrix} g(L)\ (I-\bar{J}L)^a & \\ & (1-L)^m\ (I-\hat{J}L)^a \end{pmatrix} \begin{pmatrix} \bar{q} \\ \hline \hat{q} \end{pmatrix}$$

$$= g(L)\ \bar{p}\ (I-\bar{J}L)^a\ \bar{q}\ + (1-L)^m\ \hat{p}\ (I-\hat{J}L)^a\ \hat{q}$$

Moreover $\bar{J} = I + N$, with $N^d = 0$ and $I - \bar{J}L = (1-L)\ I - NL$. This implies

$$(I-\bar{J}L)^a = (1-L)^{m-d}\ [(1-L)^{d-1}\ I + (1-L)^{d-2}\ NL + \dots + (NL)^{d-1}]$$

Indeed we can check that this formula implies

$$(I-\bar{J}L)\quad (I-\bar{J}L)^a = (1-L)^m\ I = \det\ (I-\bar{J}L)$$

This proves part i) of proposition 2.

We must prove that

$$\alpha\ \bar{p}\ N^{d-1}\ \bar{q} = \alpha\ \bar{p}\ N^{d-2}\ \bar{q} = \ldots = \alpha\ \bar{p}\ N^{d-i}\ \bar{q} = 0$$

is equivalent to $\alpha$ being orthogonal to $F_i$. The proposition has been proved for i=1 in the text. Let us assume the proposition to be true for i-1 and show that it holds for i.

We shall use the fact that the only non-zero entries of $N_k{}^h$ are those such that the difference between the column and row indices is equal to h.

$$\bar{p}\ N^{d-i}\ \bar{q} = \sum_{s\in\sigma_d} p_s N_d{}^{d-i} q_s + \sum_{s\in\sigma_{d-1}} p_s N_{d-1}^{d-i} q_s + \ldots + \sum_{s\in\sigma_{d-i+1}} p_s N_{d-i+1}^{d-i} q_s$$

$$= \sum_{s\in\sigma_d} \left(v_{s\,1}\ w_{s\ i_s - i+1} + \ldots + v_{s\,i}\ w_{s\,i_s}\right)$$

$$+ \sum_{s\in\sigma_{d-1}} (v_{s\,1}\ w_{s\ i_s - i+2} + \ldots + v_{s\ i-1}\ w_{s\,i_s}) + \ldots + \sum_{s\in\sigma_{d-i+1}} v_{s\,1}\ w_{s\,i_s}$$

Note that the vectors $v_{s\,i}$ which appear in this formula span subspace $F_i$. Let us define a matrix $\tilde{P}_i$ the column of which are the vectors which span $F_i$. Then we have $\bar{p}\ N^{d-i}\ \bar{q} = \tilde{P}_i\ \tilde{Q}_i$ with an appropriate matrix $\tilde{Q}_i$.

Let $\alpha$ be a row vector such that $\alpha\ H(L)$ has a unit root of order i. If the proposition is true for i-1, $\alpha$ is orthogonal to all vectors in $F_{i-1}$ and therefore to $(v_{s\,1}, v_{s\,2}, \ldots, v_{s\ i-1})$ for $s \in \sigma_d$, to $(v_{s\,1}, v_{s\,2}, \ldots, v_{s\ i-2})$ for $s \in \sigma_{d-1}$, ... , to $v_{s\,1}$ for $s \in \sigma_{d-i+2}$. Therefore

$$\alpha\ \bar{p}\ N^{d-i} \bar{q} = \sum_{s\in\sigma_d} (\alpha\ v_{s\,i})\ w_{s\,i_s} + \sum_{s\in\sigma_{d-1}} (\alpha\ v_{s\ i-1})\ w_{s\,i_s}$$

$$+ \dots + \sum_{s \in \sigma_{d-i+1}} (\alpha\, v_{s\,1}) w_{s i_s}$$

As all vectors $w_{s\, i_s}$ are independent eigenvectors $\alpha\, \bar{p}\, N^{d-i}\, \bar{q} = 0$ implies that all scalars $\alpha\, v_{s\,1}$, $s \in \sigma_d$, $\alpha\, v_{s\ i-1}$, $s \in \sigma_{d-1}$, ..., $\alpha\, v_{s\,1}$, $s \in \sigma_{d-i+1}$ be zero. This proves that $\alpha$ is orthogonal to all vectors of $F_i$ not belonging to $F_{i-1}$.

**Notes**

* This paper is a revised version of d'Autume(1990) "Cointegration of Higher Orders, A Clarification", Doc n° 90-23, DELTA.

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